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1998 J. Phys. A: Math. Gen. 31 9761

(http://iopscience.iop.org/0305-4470/31/48/010)

J. Phys. A: Math. Gen. **31** (1998) 9761–9768. Printed in the UK PII: S0305-4470(98)94647-9

Determination of surfaces using Beltramians at the separation of two fluid media

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Received 29 May 1998, in final form 14 September 1998

Abstract. This paper describes a model formed of a set of elliptic partial differential equations involving Beltramians of coordinates for generating the curved surfaces at the separation of two different media. The main point is that the differential equations are formed in such a way that the sum of the local principal curvatures appears explicitly, which in turn depends on the physical and chemical properties of the two media. This dependence is quite obvious as far as the problem of surface tension is concerned. The developed partial differential equations have been solved numerically for the simple case of a fluid in contact with a vertical wall and for generation of the surface of revolution due to the capillary effect in a tube of small radius. These examples demonstrate the application of the method when the surface tension is important. The method may be of value in tackling other problems, such as the generation of synclastic and anticlastic surfaces, provided that the local principal curvatures can be related to the coordinates by using some physical principle.

1. Introduction

The surface phenomenon occurring near the surface of separation of two continuous media depends on the physical properties of the media. If the media are in mechanical and thermodynamic equilibrium then the resulting surface of separation can be determined by solving a set of second-order partial differential equations (PDEs). These PDEs explicitly contain some differential-geometric terms which have a direct bearing on the surface phenomena. The motivation of this work is essentially due to the fact that the surface-generating or shape-determination equations, i.e. the PDEs, explicitly depend on the local principal curvatures of the surface and, as shown in Batchelor [1], Case [2], Landau and Lifshitz [3], Cottrell [4], and Defay and Prigogine [5], their sum is directly related to the surface tension.

The differential equations to be presented in this paper were developed by Warsi [6–8] for an entirely different purpose. The purpose then was to have a set of model equations for generating a desired coordinate system in a given surface of arbitrary shape. However, a closer study of these equations reveals that the same equations can be used to generate a surface if the first and second fundamental forms of a surface satisfying the Mainardi–Codazzi equations are given. In fact, the same equations with a slight modification can be used to generate hypersurfaces in a four-dimensional manifold [9].

In this paper a concise description of the governing equations which automatically involve the Beltramians of surface coordinates is given. A reduced form of these equations

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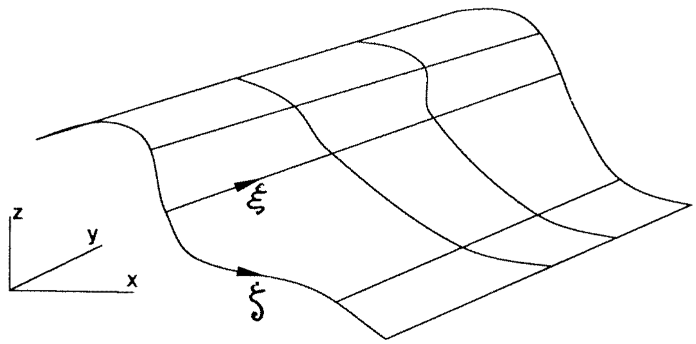

Figure 1. Determination of the shape surface $z = f(\zeta, \xi)$.

is then used to consider the case of a fluid surface in surface tension as discussed by Landau and Lifshitz [3, p 235] and the case of the surface of revolution of the meniscus for a liquid in a capillary tube of circular cross section. A comparison of the numerical and analytic solutions for the first case is shown in figure 2. The PDEs stated here can also be used in other areas of shape determination when some differential-geometric properties have been specified.

2. Analysis and applications

Let $x^i, i = 1, 2, 3$, be a right-handed curvilinear coordinate system in a three-dimensional Euclidean space of Cartesian coordinates $\boldsymbol{r} = (x, y, z)$. For simplicity of exposition we take $x^2 = \eta =$ constant as the surface, in which the coordinates are $x^3 = \zeta$ and $x^1 = \xi$. By using the formulae of Gauss (e.g. Kreyszig [10]), Warsi [7] obtained the following PDEs:

$$D\boldsymbol{r} + G_2\left[(\Delta_2\zeta)\frac{\partial \boldsymbol{r}}{\partial \zeta} + (\Delta_2\xi)\frac{\partial \boldsymbol{r}}{\partial \xi}\right] = \boldsymbol{n}R \tag{1}$$

where

$$\begin{aligned} D &= g_{11}\frac{\partial^2}{\partial \zeta^2} - 2g_{31}\frac{\partial^2}{\partial \zeta\,\partial \xi} + g_{33}\frac{\partial^2}{\partial \xi^2} \\ G_2 &= g_{11}g_{33} - (g_{31})^2 \\ R &= G_2(k_{\mathrm{I}} + k_{\mathrm{II}}) \\ \boldsymbol{n} &= (X, Y, Z) = \text{unit outward drawn normal.} \end{aligned} \tag{2}$$

Furthermore,

$$\begin{aligned} \Delta_2\zeta &= -\frac{1}{G_2}(g_{33}\Upsilon^3_{11} - 2g_{31}\Upsilon^3_{31} + g_{11}\Upsilon^3_{33}) \\ \Delta_2\xi &= -\frac{1}{G_2}(g_{33}\Upsilon^1_{11} - 2g_{31}\Upsilon^1_{31} + g_{11}\Upsilon^1_{33}) \end{aligned} \tag{3}$$

are the Beltramians of the coordinates ζ and ξ, respectively.

In (2) and (3), $g_{\alpha\beta}$ $(\alpha, \beta, = 3, 1)$ are the metric coefficients and ${}'\Upsilon^\delta_{\alpha\beta}$ $(\delta = 3, 1, \alpha, \beta = 3, 1)$ are the surface Christoffel symbols [11]. The important thing to note here is the explicit appearance of k_{I} and k_{II} which are the local principal curvatures, and the sum $k_{\mathrm{I}} + k_{\mathrm{II}}$ is invariant (except for sign) to coordinate transformation. It has been shown by Warsi [8] that the set of equations in (1) also satisfy the equations of Weingarten (e.g. Kreyszig [10]).

From this we conclude that equation (1) is free of empiricism and should be applicable to all cases of surface generation which are representable by second-order continuously differentiable functions $\boldsymbol{r} = (x(\zeta, \xi), y(\zeta, \xi), z(\zeta, \xi))$. Three distinct cases can be studied by using equation (1).

(i) Suppose the surface is given in the form $F(x, y, z) = 0$, or, $z = f(x, y)$, then $k_{\mathrm{I}} + k_{\mathrm{II}}$ can be expressed as a function of x, y, z. Furthermore, the Beltramians $\Delta_2\zeta$ and $\Delta_2\xi$ can be chosen arbitrarily, including $\Delta_2\zeta = 0$ and $\Delta_2\xi = 0$, to have a desired distribution of the coordinate curves ζ and ξ. This is the problem of surface grid generation where the parametric space (ζ, ξ) can be mapped to the integer space (K, I).

(ii) If the coefficient of the first and the second fundamental forms $g_{\alpha\beta}$ and $b_{\alpha\beta}$, respectively, are given and satisfy the Mainardi–Codazzi equations then a surface can be generated by solving equation (1), which is unique except for its position in space (cf Warsi [9]). In this case

$$k_{\mathrm{I}} + k_{\mathrm{II}} = g^{\alpha\beta} b_{\alpha\beta} \tag{4}$$

where $g^{\alpha\beta}$ are the contravariant components of the metric tensor and

$$g^{\alpha\gamma} g_{\beta\gamma} = \delta^{\alpha}_{\beta}$$

with the repeated indices implying a sum. Equation (4) is invariant to coordinate transformation.

(iii) If $k_{\mathrm{I}} + k_{\mathrm{II}}$ is a known function of $\boldsymbol{r} = (x, y, z)$ then a surface can be generated by solving equation (1) under appropriate boundary conditions.

The subject matter of this paper is item (iii). All cases (i)–(iii) require a numerical method for the solution of the PDEs. Since the equations are elliptic and quasilinear, the method of successive-over-relaxation (SOR) with iteration seems to work well for all cases. If the problem can be posed as an initial-value problem, then the Runge–Kutta method works quite well.

To demonstrate the use of the developed equation, we consider the problem of surface tension between two media, denoted by 1 and 2, which are separated by a curved surface. Furthermore, we assume that the normal $\boldsymbol{n}$ to the surface is directed in medium 1. Following the results obtained by Batchelor [1], the relation between the sum of the principal curvatures and the stress tensor can be put in the form

$$-\alpha(k_{\mathrm{I}} + k_{\mathrm{II}})\boldsymbol{n} = (\boldsymbol{T}_1 - \boldsymbol{T}_2) \cdot \boldsymbol{n} \tag{5}$$

where $\boldsymbol{T}$ is the stress tensor, which according to Stokes' law is

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{\sigma}. \tag{6}$$

In (6) p is the pressure, $\boldsymbol{\sigma}$ is the deviatoric stress tensor and $\boldsymbol{I}$ is the unit tensor. In (5), the subscripts 1 and 2 on $\boldsymbol{T}$ denote the respective medium, and $\boldsymbol{\alpha}$ is the surface-tension coefficient. Substituting (6) in (5), we obtain

$$\alpha(k_{\mathrm{I}} + k_{\mathrm{II}})\boldsymbol{n} = (p_1 - p_2)\boldsymbol{n} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{n} + \boldsymbol{\sigma}_2 \cdot \boldsymbol{n}. \tag{7}$$

Equation (7) is valid for all cases, namely, synclastic (both centres of curvature lie on the same side), or anticlastic (centres lie on opposite sides) surfaces. In a gravitational field and neglecting the effect of viscosity, equation (7) reduces to Laplace's formula

$$\boldsymbol{\alpha}(k_{\mathrm{I}} + k_{\mathrm{II}}) = p_1 - p_2. \tag{8}$$

A large class of shape formations due to surface tension have the form $z = f(x, y)$. In such cases taking

$$x = \zeta \qquad y = \xi \qquad z = f(\zeta, \xi)$$

the Beltramians as stated in equations (3) are

$$\Delta_2\zeta = \Delta_2 x = -\frac{z_x}{G_2^2}Dz$$
$$\Delta_2\xi = \Delta_2 y = -\frac{z_y}{G_2^2}Dz.$$

Furthermore,

$$g_{11} = 1 + z_y^2 \qquad g_{31} = g_{13} = z_x z_y \qquad g_{33} = 1 + z_x^2$$
$$Dz = (1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy}$$
$$G_2 = 1 + z_x^2 + z_y^2$$

and the components of the normal $\boldsymbol{n}$ are

$$X = -z_x/G_2^{1/2} \qquad Y = -z_y/G_2^{1/2} \qquad Z = 1/G_2^{1/2}.$$

Thus all three equations from the vector equation (1) reduce to a single scalar equation

$$Dz - G_2^{3/2}(k_{\mathrm{I}} + k_{\mathrm{II}}) = 0$$

or

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = (1 + z_x^2 + z_y)^{3/2}(k_{\mathrm{I}} + k_{\mathrm{II}}) \tag{9}$$

where a variable subscript denotes a partial derivative. A computer program to solve equation (9) for a given function $k_{\mathrm{I}} + k_{\mathrm{II}}$ under the appropriate boundary condition is available to determine the shape function. The above discussion is summarized by the sketch in figure 1.

As an application of the equations developed above, we consider the fluid surface in a gravitational field with medium 2 as a fluid and medium 1 as air. Thus,

$$p_1 = p_{\mathrm{atm}} \qquad p_2 = \text{constant} - \rho g z$$

where ρ is the fluid density and z is measured vertically upwards. Thus,

$$p_1 - p_2 = \rho g z + \text{constant}$$

and from (8)

$$k_{\mathrm{I}} + k_{\mathrm{II}} = \frac{2z}{a^2} + \text{constant} \tag{10}$$

where $a = \left(\frac{2\alpha}{\rho g}\right)^{1/2}$ is the capillary constant. For the case of a fluid surface in a gravitational field and bounded on one side by a wall parallel to the z-coordinate (Landau and Lifshitz [3, p 235]), since both k_{I} and k_{II} vanish when $z = 0$, the constant appearing in (10) is zero. Furthermore, let h be the maximum value of z where the fluid attaches to the wall. Non-dimensionalizing all the quantities in equation (9) by h and using the same symbols, the governing equation becomes

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 2\frac{h^2}{a^2}(1 + z_x^2 + z_y^2)^{3/2}z. \tag{11}$$

The appropriate boundary conditions are

$$z(0, y) = 1 \qquad z(\infty, y) = 0$$
$$z_x(0, y) = -\cot\theta \qquad z_x(\infty, y) = 0$$
$$z_y(x, 0) = 0 \qquad z_y(x, \infty) = 0$$

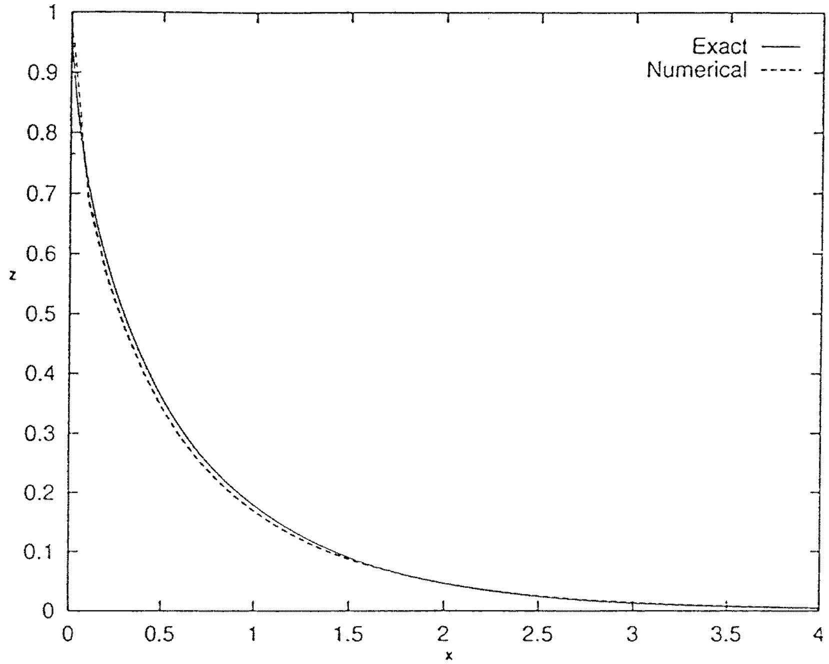

Figure 2. Trace of the surface in the xz-plane. The surface is formed due to surface tension with a wall on the left along the z-axis and free on the right. The contact height is h and the contact angle is $\theta = 5°$. The surface extends along the y-axis, and $z = 0$ at $x = \infty$. In this figure x and z are non-dimensional and $\Delta x = 0.1$, $\Delta y = 0.04$.

where θ is the angle of contact at $z = 1$, and

$$\frac{h^2}{a^2} = 1 - \sin\theta.$$

Landau and Lifshitz [3] have further simplified equation (11) by taking $k_{\mathrm{I}} = 0$ and $k_{\mathrm{II}} = (1 + z_x^2)^{3/2}/z_{xx}$ and obtained the exact solution in non-dimensional variables as†

$$x = \frac{a}{h\sqrt{2}} \ln\left[\frac{\sqrt{2}a}{hz} + \left(\frac{2a^2}{h^2z^2} - 1\right)^{1/2}\right] - \frac{a}{h}\left(2 - \frac{h^2z^2}{a^2}\right)^{1/2} + x_0 \qquad (12)$$

where

$$x_0 = \frac{a}{h}\left(2 - \frac{h^2}{a^2}\right)^{1/2} - \frac{a}{h\sqrt{2}} \ln\left[\frac{\sqrt{2}a}{h} + \left(\frac{2a^2}{h^2} - 1\right)^{1/2}\right].$$

Since the purpose of this paper is to demonstrate the use of equation (1) and hence of equation (11), we have solved the quasilinear equation (11) by the line SOR using iteration. It may be pointed out here that in the numerical solution of equation (11) the condition $k_{\mathrm{I}} = 0$ has not been used. The numerical solution so obtained is compared with (12) and shown in figure 2. Since the SOR uses only a second-order difference approximation of the derivatives in equation (11) some discrepancy occurs between the numerical and exact solution.

† Note that the correct solution is the negative of the solution in Landau and Lifshitz [3].

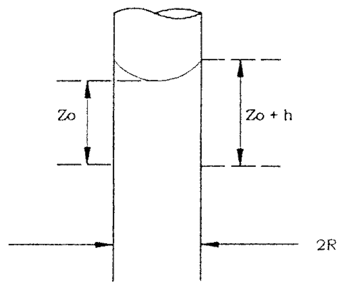

Figure 3. Sketch showing the meniscus in a vertical capillary tube of circular cross section.

As a second application of the method, we consider the problem of the generation of surface of revolution of the meniscus for a liquid in a capillary tube of circular cross section. Again taking the height of the meniscus h as a non-dimensionalizing length, we first define

$$r = (x^2 + y^2)^{1/2}$$

and then using the chain rule of differentiation, equation (9) reduces to

$$z_{rr} + \frac{1}{r}(1 + z_r^2)z_r = (1 + z_r^2)^{3/2}(k_{\mathrm{I}} + k_{\mathrm{II}}). \tag{13}$$

Case [2] has solved equation (13) by neglecting z_r^z and using the modified Bessel functions. Here we propose to solve the complete equation (13). Furthermore, from [2]

$$k_{\mathrm{I}} + k_{\mathrm{II}} = \frac{2h^2}{a^2}\left(1 - \frac{\rho_a}{\rho}\right)z \tag{14}$$

where, as used earlier,

$$a = \left(\frac{2\alpha}{\rho g}\right)^{1/2}$$

(ρ is the density of the liquid and ρ_a the density of air). Referring to figure 3 and using the simple formula given, e.g. in White [12], we obtain

$$\frac{h^2}{a^2} = \frac{\cos\theta}{R(z_0 + 1)}$$

where θ is the angle of contact which is usually very small. The final equation to be solved is then

$$z_{rr} + \frac{1}{r}(1 + z_r^2)z_r = \frac{2\cos\theta}{R(z_0 + 1)}\left(1 - \frac{\rho_a}{\rho}\right)(1 + z_r^2)^{3/2}z. \tag{15}$$

The initial conditions are

$$z(0) = z_0 \qquad z_r(0) = 0. \tag{16}$$

It must be restated here that all quantities in (15) and (16) are non-dimensional. Equation (15) under the initial conditions has been solved by forming the system of first-order equations:

$$\begin{aligned}
z_r &= f \\
f_r &= \frac{2\cos\theta}{R(z_0 + 1)}\left(1 - \frac{\rho_a}{\rho}\right)(1 + f^2)^{3/2}z - \frac{1}{r}(1 + f^2)f \\
z(0) &= z_0 \\
f(0) &= 0
\end{aligned}$$

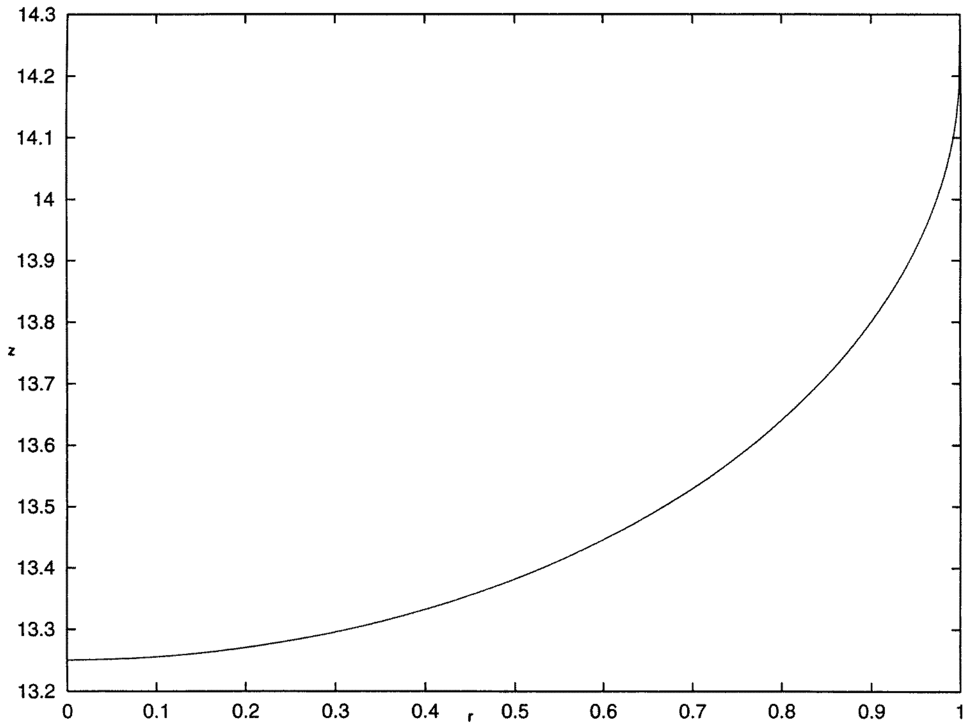

Figure 4. Cross section of the surface of revolution $z(r)$ of the meniscus in a vertical capillary tube of radius R. All quantities are non-dimensional. The surface is generated by rotating the curve $z(r)$ about the z-axis.

and using a fourth-order Runge–Kutta method with a step size $\Delta r = 0.001$. The following data has been used. From [12] the dimensional quantities are taken as

$$z_0 + h = 15 \text{ mm} \qquad R = 1 \text{ mm} \qquad \alpha = 0.073 \text{ Nm}^{-1} \qquad \rho = 1\,000 \text{ kg m}^{-3}.$$

Thus

$$a^2 = \frac{2\alpha}{\rho g} = 0.000\,014\,883 \text{ m}^2.$$

Though the purpose is to demonstrate the generation of a surface, nevertheless, we have taken the appropriate value of h and hence of z_0 obtained by solving the equation $z(1; z_0) = z_0 + 1$. Thus

$$h = 1.052\,579\,873 \text{ mm}$$

so that

$$z_0 = 13.947\,420\,13 \text{ mm}.$$

The dimensionless quantities are

$$z_0 = 13.2507$$
$$R = 0.950\,046\,667$$
$$2\left(1 - \frac{\rho_a}{\rho}\right) = 1.997\,546\,392$$
$$\theta = 1^\circ.$$

The solution curve obtained by solving (15) and (16) is shown in figure 4. Rotation of this curve about the z-axis is the required surface of revolution.

3. Conclusion

Surface phenomena play an important role in many physical processes. The main conclusion of this paper is to show that if the local principal curvatures of the separation surface of the two media can be related to the properties of the media then the separation surface can be generated. This is accomplished by a set of elliptic PDEs involving the Beltramians of coordinates and in which the sum of the principal curvatures appear explicitly, i.e. equation (1). In the case of the problem of surface determination under surface tension, Laplace's formula connects the sum of the principal curvatures with the pressure and viscous forces. This paper uses the above-mentioned relation and generates the surface formed due to surface tension in two cases. The numerical result for a fluid resting against a vertical wall is compared with the available exact solution. It may be restated here that the Cauchy data to solve equation (11) is used to solve a Dirichlet problem by iteration. In other cases, equation (1) has to be solved for a prescribed function R imposing the proper boundary conditions.

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